Noether's Theorem

From the symplectic perspective

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Claim

Suppose we have a symplectic manifold (M, ω) *and a Lie group* G *which acts on* M *by symplectomorphism with momentum mapping* $J : M \to \mathfrak{g}^*$*. Suppose* $H : M \to \mathbb{R}$ *is invariant under the action. Then,* J *is constant on integral lines of* X_H*.*

Preliminaries

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  We explain:
- Symplectic forms and Symplectomorphisms,
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- and Hamiltonian vector fields.

Symplectic Forms

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- ω is closed (i.e., $d\omega = 0$)
- ω is nondegenerate. i.e. if $\omega(u, v) = 0$ for all $v \in T_pM$, then $u = 0$.

Symplectomorphisms

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- i.e. $\left(f^*\omega\right)_p(X_1, X_2) = \omega_{f(p)}\left(df_p(X_1), df_p(X_2)\right) = \omega_p(X_1, X_2)$

The Momentum Mapping

Suppose G acts on (M, ω) by symplectomorphism.

Then $J : M \to \mathfrak{g}^*$ is called the momentum mapping for G if for all $\xi \in \mathfrak{g}$,

- $dJ(x) \cdot \xi = i_{\xi_M}\omega := \omega\left(\xi_M, x\right)$

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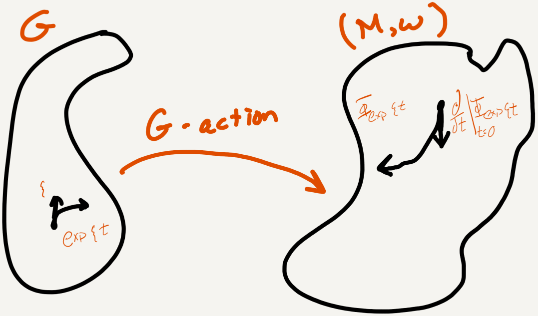
Then $J : M \to \mathfrak{g}^*$ is called the momentum mapping for G if for all $\xi \in \mathfrak{g}$,

- $dJ(x) \cdot \xi = i_{\xi_M}\omega := \omega\left(\xi_M, x\right)$
  Which is equivalent to
- $X_{J(x)\cdot\xi} = \xi_M \equiv \frac{d}{dt}\left(\exp t\xi, x\right)\big|_{t=0}$



Hamiltonians

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- Then, the vector field X_H is given by

$$i_{X_H}\omega = \omega\left(X_H, -\right) = dH(-)$$

- X_H is guaranteed to exist by the nondegeneracy of ω.

Constancy Lemma

Claim

Take $f, g : M \to \mathbb{R}$. f is constant on integral curves of X_g iff g is constant on integral lines of X_f iff $\{f, g\} = 0$

Constancy Lemma: Proof

Recall that $\{f, g\}$ denotes the Poisson bracket of f and g;
$\{f, g\} = -i_{X_f} i_{X_g} \omega = -L_{X_f} g$
Suppose F_t is the flow of X_f, and that g is constant on it.

- $0 = \frac{d}{dt}(g \circ F_t) = \frac{d}{dt}(F_t^* g) = F_t^* L_{X_f} g = -F_t^* \{f, g\}$

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- $F_t^* \{f,g\} = 0$ for all t occurs iff $\{f,g\} = 0$

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Let F_t be the flow of X_H. If $\xi \in \mathfrak{g}$, $H\left(\Phi_{\exp t\xi}(x)\right) = H(x)$ by the invariance of H.

Differentiating wrt t at $t = 0$, we obtain

- $dH(x) \cdot \xi_M = 0$

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  By the Poisson bracket,
- $\{H, J(x) \cdot \xi\} = 0$

Noether's Theorem: Proof Continued

By the constancy lemma,

- $\frac{d}{dt}\left(J(F_t(x)) \cdot \xi\right) = 0$
  By the property of flow, $F_0(x) = x$. Thus,
- $J(F_t(x)) \cdot \xi = J(F_0(x)) \cdot \xi = J(x) \cdot \xi$

And so J is constant on integral curves of X_H, because H is preserved by the group action.