Symplectic Geometry and the Uncertainty Principle

Ryan McWhorter

Department of Mathematics
The University of Texas at Austin

February 12, 2023

The Uncertainty Principle

Claim

The Schrödinger-Robertson inequality

$$\Delta X^2 \Delta P^2 \geq \mathrm{Cov}\left(X, P\right)^2 + \frac{1}{4}\hbar^2$$

can be considered a consequence of symplectic geometry if covariances are treated as measurement error.

Disclaimer

- We are *not* claiming that quantum mechanics is broadly a consequence of symplectic geometry.

Disclaimer

- We are *not* claiming that quantum mechanics is broadly a consequence of symplectic geometry.
- We're just giving a limited taste of it in a classical setting.

Preliminaries

- We assume some foreknowledge of differential geometry (manifolds, the Lie/exterior derivative, differential forms, flows, etc)

Preliminaries

- We assume some foreknowledge of differential geometry (manifolds, the Lie/exterior derivative, differential forms, flows, etc)
  We explain:
- Symplectic forms and symplectomorphisms,

Preliminaries

- We assume some foreknowledge of differential geometry (manifolds, the Lie/exterior derivative, differential forms, flows, etc)
  We explain:
- Symplectic forms and symplectomorphisms,
- and Hamiltonian flows (which is the bedrock of classical mechanics).

Symplectic Forms

A differential 2-form ω is called a symplectic form if

- ω is closed (i.e., $d\omega = 0$)

Symplectic Forms

A differential 2-form ω is called a symplectic form if

- ω is closed (i.e., $d\omega = 0$)
- ω is nondegenerate. i.e. if $\omega(u, v) = 0$ for all $v \in T_pM$, then $u = 0$.

Symplectomorphisms

$f : M \to M$ is called a symplectomorphism if $f^*\omega = \omega$.

Symplectomorphisms

$f : M \to M$ is called a symplectomorphism if $f^*\omega = \omega$.

- i.e. $\left(f^*\omega\right)_p (X_1, X_2) = \omega_{f(p)} \left(df_p(X_1), df_p(X_2)\right) = \omega_p(X_1, X_2)$

Hamiltonians

Suppose we have $H : M \to \mathbb{R}$.

Hamiltonians

Suppose we have $H : M \to \mathbb{R}$.

- Then, the vector field X_H is given by

$$i_{X_H}\omega = \omega\left(X_H, -\right) = dH(-)$$

- X_H is guaranteed to exist by the nondegeneracy of ω.

Hamiltonians

Suppose we have $H : M \to \mathbb{R}$.

- Then, the vector field X_H is given by

$$i_{X_H}\omega = \omega\left(X_H, -\right) = dH(-)$$

- X_H is guaranteed to exist by the nondegeneracy of ω.
- The flow of a Hamiltonian is simply the flow of this Hamiltonian vector field

Phase Space

Where does physics happen?

Phase Space

Where does physics happen?
Phase space

Phase Space

Where does physics happen?
Phase space

- In physics, we usually take the space of positions to be $\mathbb{R}^3$.

Phase Space

Where does physics happen?
Phase space

- In physics, we usually take the space of positions to be $\mathbb{R}^3$.
- The phase space of $\mathbb{R}^3$ is $\left(T^*\mathbb{R}^3, \sum_{i=1}^{3} dp_i \wedge dq_i\right)$

Phase Space

Where does physics happen?
Phase space

- In physics, we usually take the space of positions to be $\mathbb{R}^3$.
- The phase space of $\mathbb{R}^3$ is $\left(T^*\mathbb{R}^3, \sum_{i=1}^{3} dp_i \wedge dq_i\right)$
- Generically, the phase space of $\mathbb{R}^n$ is $(T^*\mathbb{R}^n, \sum_{i=1}^{n} dp_i \wedge dq_i)$

Symplectic Capacities

What are they and why are they useful?

Symplectic Capacities

What are they and why are they useful?

- A symplectic capacity assigns a non-negative real number to an arbitrary region of $\mathbb{R}^{2n}$

Symplectic Capacities

What are they and why are they useful?

- A symplectic capacity assigns a non-negative real number to an arbitrary region of $\mathbb{R}^{2n}$
- Capacities give us a way to determine if one subspace *doesn't* symplectically embed into another

Symplectic Capacities — Definition

Suppose c is our capacity.

- Monotonicity: if there exists a symplectic embedding $\phi : A \hookrightarrow \mathbb{R}^{2n}$, such that $\phi(A) \subset B$ $c\,(A) \leq c\,(B)$.

Symplectic Capacities — Definition

Suppose c is our capacity.

- Monotonicity: if there exists a symplectic embedding $\phi : A \hookrightarrow \mathbb{R}^{2n}$, such that $\phi(A) \subset B$ $c(A) \leq c(B)$.
- Conformality: for all $\lambda \in \mathbb{R}^{+}$, $c(\lambda A) = \lambda^2 c(A, \omega)$

Symplectic Capacities — Definition

Suppose c is our capacity.

- Monotonicity: if there exists a symplectic embedding $\phi : A \hookrightarrow \mathbb{R}^{2n}$, such that $\phi(A) \subset B$ $c(A) \leq c(B)$.
- Conformality: for all $\lambda \in \mathbb{R}^+$, $c(\lambda A) = \lambda^2 c(A, \omega)$
- Non-triviality: $c\left(B^{2n}(1), \omega_0\right) > 0$ and $c\left(Z^{2n}(1), \omega_0\right) < \infty$. (Here $\left(Z^{2n}(1), \omega_0\right)$ is a cylinder of radius 1).

Symplectic Capacities — Definition

Suppose c is our capacity.

- Monotonicity: if there exists a symplectic embedding $\phi : A \hookrightarrow \mathbb{R}^{2n}$, such that $\phi(A) \subset B$ $c\,(A) \leq c\,(B)$.
- Conformality: for all $\lambda \in \mathbb{R}^{+}$, $c\,(\lambda A) = \lambda^2 c\,(A, \omega)$
- Non-triviality: $c\left(B^{2n}(1), \omega_0\right) > 0$ and $c\left(Z^{2n}(1), \omega_0\right) < \infty$. (Here $\left(Z^{2n}(1), \omega_0\right)$ is a cylinder of radius 1).
  This last requirement is really difficult to satisfy! As a result, there are very few symplectic capacities.

Symplectic Capacities — Gromov's Width

An example of a capacity:

$$c_{gw}\left(M,\omega\right)=c_{gw}\left(M\right)=\sup\{\pi r^{2}|B^{2n}\left(r\right)\text{ embeds symplectically into }M\}$$

Gromov's Non-Squeezing Theorem

Theorem

If there exists a symplectic embedding of $\left(B^{2n}\left(r\right),\omega_0\right)$ *into* $\left(Z^{2n}\left(R\right),\omega_0\right)$*, then* $r \leq R$*.*

Gromov's Non-Squeezing Theorem

Theorem

If there exists a symplectic embedding of $\left(B^{2n}\left(r\right),\omega_0\right)$ *into* $\left(Z^{2n}\left(R\right),\omega_0\right)$*, then* $r \leq R$*.*

- This is equivalent to proving that Gromov's width satisfies the symplectic capacity axioms.

Gromov's Non-Squeezing Theorem

Theorem

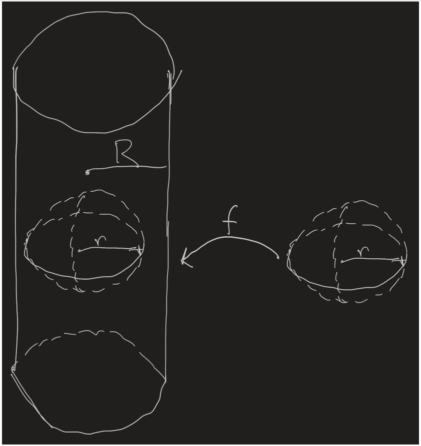
If there exists a symplectic embedding of $\left(B^{2n}\left(r\right),\omega_0\right)$ *into* $\left(Z^{2n}\left(R\right),\omega_0\right)$*, then* $r \leq R$*.*

- This is equivalent to proving that Gromov's width satisfies the symplectic capacity axioms.
- We don't provide proof of Gromov's theorem/proof of the existence of Gromov's width - that's beyond the scope of this presentation.

Gromov's Non-Squeezing Theorem — Illustration

Theorem

If there exists a symplectic embedding of $\left(B^{2n}\left(r\right), \omega_0\right)$ *into* $\left(Z^{2n}\left(R\right), \omega_0\right)$*, then* $r \leq R$*.*



Back to the Uncertainty Principle

Recall:

Claim

The Schrödinger-Robertson inequality

$$\Delta X^2 \Delta P^2 \geq \mathrm{Cov}\left(X, P\right)^2 + \frac{1}{4}\hbar^2$$

can be considered a consequence of symplectic geometry if covariances are treated as measurement error.

What is the Uncertainty Principle

- This is usually phrased as the fact that there is a fundamental physical limit on how precisely you can measure position and momentum, and that there is a trade off between precision of measuring position and precision of momentum.

Setting Up the Uncertainty Principle

Suppose we start out with a bunch measurements of a physical system in $(T^*\mathbb{R}^n, \sum_{i=1}^n dp_i \wedge dq_i)$.

Setting Up the Uncertainty Principle

Suppose we start out with a bunch measurements of a physical system in $(T^*\mathbb{R}^n, \sum_{i=1}^n dp_i \wedge dq_i)$.

- First, we want to find an ellipsoid $\mathcal{J}$ that contains these points with the smallest volume. $\mathcal{J}$ is of the form $(z - \bar{z})^T M^{-1} (z - \bar{z}) \leq m_0^2$ for some M and some m_0.

Setting Up the Uncertainty Principle

Suppose we start out with a bunch measurements of a physical system in $(T^*\mathbb{R}^n, \sum_{i=1}^n dp_i \wedge dq_i)$.

- First, we want to find an ellipsoid $\mathcal{J}$ that contains these points with the smallest volume. $\mathcal{J}$ is of the form $(z - \bar{z})^T M^{-1} (z - \bar{z}) \leq m_0^2$ for some M and some m_0.
- m_0^2 is determined by the distribution of observations in phase space. For normally distributed points, $m_0 = \chi^2_{0.5}(2n)$

Capacity of an Ellipsoid

- In order to derive the uncertainty principle, we're going to have compute the capacity of an ellipsoid.

Capacity of an Ellipsoid

- In order to derive the uncertainty principle, we're going to have compute the capacity of an ellipsoid.
- Capacities don't in general agree, but they do in the case of phase space ellipsoids (we won't show why though).

Capacity of an Ellipsoid: Derivation

Let c_{gw} be Gromov's width, and let Ω_{ell} be the ellipsoid $z^T M z \leq 1$ centered at $\bar{z} = 0$.

- Capacities are invariant under symplectomorphisms, so we can center our ellipsoid at $\bar{z} = 0$

Capacity of an Ellipsoid: Derivation

Let c_{gw} be Gromov's width, and let Ω_{ell} be the ellipsoid $z^T M z \leq 1$ centered at $\bar{z} = 0$.

- Capacities are invariant under symplectomorphisms, so we can center our ellipsoid at $\bar{z} = 0$
- By Williamson's diagonalization theorem, there is a symplectic matrix S such that $S^T M S = \begin{bmatrix} \Lambda & \\ & \Lambda \end{bmatrix}$ with $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n)$

Capacity of an Ellipsoid: Derivation

Let c_{gw} be Gromov's width, and let Ω_{ell} be the ellipsoid $z^T M z \leq 1$ centered at $\bar{z} = 0$.

- Capacities are invariant under symplectomorphisms, so we can center our ellipsoid at $\bar{z} = 0$
- By Williamson's diagonalization theorem, there is a symplectic matrix S such that $S^T M S = \begin{bmatrix} \Lambda & \\ & \Lambda \end{bmatrix}$ with $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n)$
- Thus, $S\left(\Omega_{ell}\right) = \sum_{i=1}^{n} \lambda_i \left(x_i^2 + y_i^2\right) \leq 1$.

Capacity of an Ellipsoid: Derivation

Let c_{gw} be Gromov's width, and let Ω_{ell} be the ellipsoid $z^T M z \leq 1$ centered at $\bar{z} = 0$.

- Capacities are invariant under symplectomorphisms, so we can center our ellipsoid at $\bar{z} = 0$
- By Williamson's diagonalization theorem, there is a symplectic matrix S such that $S^T M S = \begin{bmatrix} \Lambda & \\ & \Lambda \end{bmatrix}$ with $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n)$
- Thus, $S\left(\Omega_{ell}\right) = \sum_{i=1}^{n} \lambda_i \left(x_i^2 + y_i^2\right) \leq 1$.
- Thus, we have $c_{gw}\left(\Omega_{ell}\right) = c_{gw}\left(S\left(\Omega_{ell}\right)\right)$

Capacity of an Ellipsoid: Derivation

Let c_{gw} be Gromov's width, and let Ω_{ell} be the ellipsoid $z^T M z \leq 1$ centered at $\bar{z} = 0$.

- Suppose there's a symp. embedding sending $B(R)$ inside (Ω_{ell}).

Capacity of an Ellipsoid: Derivation

Let c_{gw} be Gromov's width, and let Ω_{ell} be the ellipsoid $z^T M z \leq 1$ centered at $\bar{z} = 0$.

- Suppose there's a symp. embedding sending $B(R)$ inside (Ω_{ell}).
- Then, there is a symp. embedding sending $B(R)$ inside the cylinder $Z\left(\sqrt{1/\lambda_i}\right) = x_i^2 + y_i^2 \leq \lambda_i$ for all eigenvalues λ_i.

Capacity of an Ellipsoid: Derivation

Let c_{gw} be Gromov's width, and let Ω_{ell} be the ellipsoid $z^T M z \leq 1$ centered at $\bar{z} = 0$.

- Suppose there's a symp. embedding sending $B(R)$ inside (Ω_{ell}).
- Then, there is a symp. embedding sending $B(R)$ inside the cylinder $Z\left(\sqrt{1/\lambda_i}\right) = x_i^2 + y_i^2 \leq \lambda_i$ for all eigenvalues λ_i.
- In light of the non-squeezing theorem, the capacity of the ellipsoid must be at least equal to the capacity of the smallest cylinder, $Z\left(1/\lambda_{\text{max}}\right)$.

Conclusion

Let's return to our minimum volume ellipsoid, $\mathcal{J}$. We can conclude two things:

- Firstly, since symplectic capacities are invariant under symplectomorphism, and since physical systems evolve under time-dependent symplectomorphisms, we have that the capacity of $\mathcal{J}$ doesn't vary as a function of time.

Conclusion

Let's return to our minimum volume ellipsoid, $\mathcal{J}$. We can conclude two things:

- Firstly, since symplectic capacities are invariant under symplectomorphism, and since physical systems evolve under time-dependent symplectomorphisms, we have that the capacity of $\mathcal{J}$ doesn't vary as a function of time.
- With some more statistics and linear algebra, you can relate this capacity of $\mathcal{J}$ into a capacity of a covariance ellipsoid, and derive the full Schrödinger-Robertson Uncertainty Principle